

1.4 Factors and Prime Factorization

Recall from Section 1.2 that the word **factor** refers to a number which divides into another number. For example, 3 and 6 are factors of 18 since $3 \cdot 6 = 18$. Note also that $2 \cdot 9 = 18$, so 2 and 9 are factors of 18 also, as are 1 and 18 ($1 \cdot 18 = 18$). The entire list of factors for 18 are thus 1, 2, 3, 6, 9, 18. Finding the factors of a number is thus a trial-and-error approach to find all numbers which divide our given number. The products (for example, $3 \cdot 6 = 18$) are called **factorizations** of 18. To find the factors of a number, it is best to proceed in an organized fashion beginning with 1, 2, 3, The following example will help to illustrate this procedure.

Example 1 Find all factors of the given number.

- a. 12
- b. 20
- c. 36
- d. 53

Solution

- a. Note the following factorizations of 12:

$$1 \cdot 12 = 12$$

$$2 \cdot 6 = 12$$

$$3 \cdot 4 = 12$$

So the factors of 12 are 1, 2, 3, 4, 6, 12.

- b. Note the following factorizations of 20:

$$1 \cdot 20 = 20$$

$$2 \cdot 10 = 20$$

$$4 \cdot 5 = 20$$

So the factors of 20 are 1, 2, 4, 5, 10, 20.

- c. Note the following factorizations of 36:

$$1 \cdot 36 = 36$$

$$2 \cdot 18 = 36$$

$$3 \cdot 12 = 36$$

$$4 \cdot 9 = 36$$

$$6 \cdot 6 = 36$$

So the factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36.

- d. The number 53 has only one factorization ($1 \cdot 53 = 53$), so its only factors are 1 and 53.

There are a few “tricks” which can help you speed up the process. First note that 2 is a factor of a number if that number is even. Also note that 3 is a factor of a number if the sum of the digits in the number is divisible by 3. For example, since $3 + 6 = 9$, and 3 is a factor of 9, then 3 is a factor of 36. This is particularly useful for large numbers such as 1,011 (since $1 + 1 + 1 = 3$, 3 is a factor of 1,011). Finally, note that 5 is a factor of a number if the number has a ones digit of 5 or 0. There are other divisibility tests, but these three are the most useful.

One important thing to notice from Example 1 is that every number n has the factorization $1 \cdot n = n$. As a consequence, every number (other than 1) has at least two factors. Note that 53 has only 1 and 53 as factors, while the other numbers have more than two factors. A number (other than 1) with only two factors is called a **prime number**, while a number with more than two factors is called a **composite number**. The number 1 (which only has one factor of 1) is neither prime nor composite.

Example 2 Determine whether each number is prime or composite.

- a. 44
- b. 77
- c. 139
- d. 141

Solution

- a. Since 44 is an even number, it is divisible by 2. Thus it has more than two factors (at least 1,2,44), so it must be composite.
- b. Since $7 \cdot 11 = 77$, 77 is divisible by both 7 and 11. Thus it has more than two factors (at least 1,7,11,77), so it must be composite.
- c. The only factors of 139 are 1 and 139. Since it has exactly two factors, 139 is a prime number.
- d. Since $1 + 4 + 1 = 6$, and 3 is a factor of 6, then 3 is a factor of 141. Thus it has more than two factors (at least 1,3,141), so it must be composite.

Given a number larger than 1, we know it is either prime (two factors, 1 and itself) or composite (more than two factors). We now turn our attention to the factors of a composite number.

Consider the factors of 36, which we found in Example 1 to be 1, 2, 3, 4, 6, 9, 12, 18, 36. Ignoring the “1 and itself” factors, the list shortens to 2, 3, 4, 6, 9, 12, 18. Note that 2 and 3 are prime. Now list the factors of the other numbers (again ignoring 1 and the number):

4: 2, 2
6: 2, 3
9: 3
12: 2, 3, 4, 6
18: 2, 3, 6, 9

Note that 2 and 3 appear frequently. The remaining numbers are 4, 6, 9, all of which are composite with 2 and 3 as factors. So our original number 36 has many factors, but only 2 and 3 are prime factors. Can we write 36 in terms of just its prime factors? Consider one factorization of 36:

$$\begin{aligned}36 &= 6 \cdot 6 \\ &= (2 \cdot 3) \cdot (2 \cdot 3) \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \\ &= 2^2 \cdot 3^2\end{aligned}$$

Note that we have written 36 as a product of its prime factors. This is called the **prime factorization** of 36. Suppose we had started with a different factorization of 36, say $36 = 4 \cdot 9$:

$$\begin{aligned}36 &= 4 \cdot 9 \\ &= (2 \cdot 2) \cdot (3 \cdot 3) \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \\ &= 2^2 \cdot 3^2\end{aligned}$$

Note that the prime factorization at the final step is identical. Let's try another one:

$$\begin{aligned}36 &= 3 \cdot 12 \\ &= 3 \cdot (2 \cdot 6) \\ &= 3 \cdot (2 \cdot 2 \cdot 3) \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \\ &= 2^2 \cdot 3^2\end{aligned}$$

Again, we have the same prime factorization. This rather remarkable fact is summarized by the following theorem**:

Fundamental Theorem of Arithmetic

Every composite number has one unique prime factorization.

** A **theorem** in mathematics is a true statement which can be proved. We are not proving this statement, however, just noting examples which hint that the statement might be true.

Example 3 Find the prime factorization of each number. If the number is prime, state so.

- a. 72
- b. 117
- c. 250
- d. 251

Solution a. First note that 72 is composite. Starting with $72 = 9 \cdot 8$, form the prime factorization:

$$\begin{aligned}72 &= 9 \cdot 8 \\ &= (3 \cdot 3) \cdot (2 \cdot 4) \\ &= (3 \cdot 3) \cdot (2 \cdot 2 \cdot 2) \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \\ &= 2^3 \cdot 3^2\end{aligned}$$

b. Note that $1 + 1 + 7 = 9$, and 3 is a factor of 9, so 3 is a factor of 117. Forming the prime factorization:

$$\begin{aligned}117 &= 3 \cdot 39 \\ &= 3 \cdot (3 \cdot 13) \\ &= 3 \cdot 3 \cdot 13 \\ &= 3^2 \cdot 13\end{aligned}$$

- c. Note that 250 has a ones digit of 0, so 5 is a factor of 250. Forming the prime factorization:

$$\begin{aligned}250 &= 5 \cdot 50 \\ &= 5 \cdot (5 \cdot 10) \\ &= 5 \cdot (5 \cdot 2 \cdot 5) \\ &= 2 \cdot 5 \cdot 5 \cdot 5 \\ &= 2 \cdot 5^3\end{aligned}$$

- d. The only factors of 251 are 1 and 251, so 251 is a prime number.

The prime factorization of a number will be used extensively throughout our work in the next two sections, as well as when working with fractions (Chapter 3).

Terminology

factor	factorization
prime number	composite number
prime factorization	theorem
Fundamental Theorem of Arithmetic	

Exercise Set 1.4

List all factors of the given number.

- | | |
|--------|---------|
| 1. 10 | 2. 15 |
| 3. 35 | 4. 40 |
| 5. 80 | 6. 100 |
| 7. 120 | 8. 150 |
| 9. 200 | 10. 250 |
| 11. 71 | 12. 89 |

Determine whether each number is prime or composite.

- | | |
|--------------------|--------------------|
| 13. 69 | 14. 71 |
| 15. 89 | 16. 91 |
| 17. 111 | 18. 169 |
| 19. 223 | 20. 247 |
| 21. 111,111 | 22. 101,001 |
| 23. 221 | 24. 271 |

Find the prime factorization of each number. If the number is prime, write prime.

- | | |
|----------------|----------------|
| 25. 20 | 26. 30 |
| 27. 42 | 28. 52 |
| 29. 60 | 30. 64 |
| 31. 140 | 32. 160 |
| 33. 210 | 34. 220 |
| 35. 242 | 36. 248 |
| 37. 231 | 38. 237 |
| 39. 249 | 40. 251 |
| 41. 264 | 42. 369 |
| 43. 484 | 44. 480 |
| 45. 495 | 46. 497 |
| 47. 377 | 48. 399 |
| 49. 351 | 50. 435 |

Exercises 51-56 are based on the following definition. A pair of numbers are called **twinprimes** if they are both prime numbers and they are only two units apart. For example, 3 and 5 are twinprimes, as are 5 and 7.

- 51.** Find the first five twinprime pairs.
- 52.** Are 51 and 53 twinprimes?
- 53.** Are 59 and 61 twinprimes?
- 54.** Are 227 and 229 twinprimes?
- 55.** Are 259 and 261 twinprimes?
- 56.** There are seven twinprime pairs between 100 and 200. List them.

Occasionally there are formulas which attempt to predict the values of prime numbers. For example, the formula $n^2 + n + 11$ produces prime numbers when $n = 1$ (13), $n = 2$ (17), $n = 3$ (23), etc.

57. Using the formula $n^2 + n + 11$, continue to evaluate when $n = 4, 5, \dots$. What is the first value of n which does not result in a prime number?
58. Find the first value of n such that the formula $n^2 + n + 17$ does not result in a prime number.
59. Find the first value of n such that the formula $n^2 + n + 29$ does not result in a prime number.
60. Find the first value of n such that the formula $n^2 + n + 31$ does not result in a prime number.

Associated with each number n is a number $\phi(n)$ which represents the number of prime numbers less than or equal to n . For example, $\phi(5) = 3$ since there are 3 primes less than or equal to 5 (namely 2,3,5). Also $\phi(15) = 6$ since there are 6 primes less than or equal to 15 (2,3,5,7,11,13).

61. Find $\phi(31)$.
62. Find $\phi(40)$.
63. Find $\phi(83)$.
64. Find $\phi(100)$.
65. Find $\phi(2) + \phi(3) + \phi(5) + \phi(7)$.
66. Find $\phi(2) + \phi(3) + \phi(4) + \phi(5) + \phi(6) + \phi(7) + \phi(8)$.

The following two questions require a good deal of thought. Try them!

67. Compute the following values:
 - $2 \cdot 3 + 1$
 - $2 \cdot 3 \cdot 5 + 1$
 - $2 \cdot 3 \cdot 5 \cdot 7 + 1$
 - $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1$
 Do you notice anything? Can you explain what happens?
68. Explain why 1 is not considered to be a prime number.
Hint: If 1 were a prime number, what would happen to prime factorizations and the Fundamental Theorem of Arithmetic?